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POINT SETS AND ALLIED CREMONA GROUPS

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It is my purpose to develop, in a series of papers of which the first has been submitted to the *Transactions of the American Mathematical Society*, some aspects of the theory of an ordered set of n discrete points in a linear projective space, S_k, of k dimensions.[1] Such a set, denoted by P_n^k, defines projectively an 'associated' set Q_n^{n-k-2} in S_{n-k-2}, the two sets being mutually related. The study of this type of association, begun by Rosanes and Sturm,[2] is continued here. If $n = 2k + 2$ the associated sets can lie in the same S_k, and in particular they may coincide in order forming thus a 'self-associated' set. In addition to projective constructions for associated and self-associated sets, certain irrational conditions for such types of association which seem to be novel are given.

The invariants of a set P_n^k are formed from the determinants of the matrix of the coördinates of the points. An invariant is required to be homogeneous and of the same degree in the coördinates of each point and to be unaltered under permutations of the points. It appears that associated sets have proportional invariants. This theory of invariants is the natural extension of the theory of binary forms if the form be regarded as the definition of a point set P_n^1. Complete systems for P_6^1 and P_6^2 are derived and the accompanying algebra is utilized to obtain explicit equations not only for the cubic surface mapped from the plane by cubic curves on P_6^2 but also for the tritangent planes and the lines of the surface.

By a certain construction the ordered set P_n^k is mapped upon a point P of a space $\Sigma_{k(n-k-2)}$. The Cremona transformation from one map P to another map P' of the same set is discussed and eventually the construction is simplified so that this transformation is linear. The $n!$ permutations of the points of P_n^k lead to $n!$ points P in Σ which are conjugate under a Cremona group $G_{n!}$ in Σ. When $k = 1$, $G_{n!}$ is the 'cross-ratio group' of Moore.[3] A set of generators of $G_{n!}$ is exhibited. The invariant spreads of $G_{n!}$ are obtained from the invariants of P_n^k. The linear system of spreads of lowest order invariant under $G_{n!}$ is derived from the irrational invariants of P_n^k of lowest weight. Associated sets determine the same group.

The form-problem of the $G_{5!}$ determined by the associated P_5^1 and P_5^2 leads to a solution of the quintic equation; that of the $G_{6!}$ determined by the associated P_6^1 and P_6^3 and by the self-associated P_6^2 to a solution of the sextic equation. These equations have been discussed from this point of view in earlier papers of the writer.[4] The principal novelty here is the use of a connection between the self-associated P_6^2 and the theta modular functions of genus two to solve the related form-problem. In general one may say that the theory of the point set furnishes a common algebraic background not only for the general equations of degree n but also, as will appear later, for such particular equations as are involved in the determination of the lines on a cubic surface and of the double tangents of a plane quartic curve.

By introducing certain conventions it is possible to establish a mutual order for the two sets of singular points of a Cremona transformation in the plane. Since ordinary corresponding point pairs also are mutually ordered the addition of any number of ordinary pairs to the sets of singular points leads to a pair of sets which are said to be 'congruent' under Cremona transformation. The conditions under which congruence can occur are fundamentally important for effective application of the Cremona transformation to geometry or analysis. These conditions are derived up to a certain natural limit. Since two sets congruent to a third are congruent to each other the projectively distinct sets in the plane which are congruent in any order with a given set P_n^2 will be mapped by an aggregate of distinct points in Σ such that the aggregate is determined by any one of its points. If the type of transformation under which the congruence occurs and if the order be given the set is uniquely determined so that the transition from one point in Σ to another is effected by a Cremona transformation. In other words the sets P_n^2 congruent in any order with a given P_n^2 are mapped in Σ by points conjugate under an 'extended' Cremona group $G_{n,2}$ which contains $G_{n!}$ as a subgroup. For $n = 6, 7, 8$ the order of $G_{n,2}$ is $6!.72$, $7!.288$, $8!.2^6.135$ respectively but for larger values of n the $G_{n,2}$ is infinite and discontinuous. The finite groups are isomorphic respectively with those of the lines on a cubic surface, the bitangents of a plane quartic curve, and the tritangent planes of a space sextic of genus four on a quadric cone.

Let us call a Cremona transformation in S_n which can be expressed as a product of the particular transformations obtained by inverting the variables a 'regular' transformation. Obviously the regular transformations are the elements of a 'regular group.' Then the definition of congruence of point sets in S_2 as given above can be generalized to include congruence under regular transformations in S_n. Again the set P_n^k defines as above an extended group $G_{n,k}$ of Cremona transformations in $\Sigma_{k(n-k-2)}$ which in general is infinite and discontinuous. Associated sets define the same group. The only finite types in this doubly infinite (for increasing n and k) series of groups are those defined by the sets mentioned above or by their associated sets. Each $G_{n,k}$ contains $G_{n-1,k}$ as a subgroup.

The Cremona transformations in S_k under which the congruence of sets P_n^k occurs will have a certain effect upon the spreads in S_k. If the order of the spread and its multiplicity at each point of P_n^k be taken as variables,[5] this effect is expressed by a linear transformation on $n + 1$ variables. Thus there is determined a group of linear transformations $g_{n,k}$ in S'_n with integer coefficients which is isomorphic with the Cremona $G_{n,k}$. In S'_n, the $g_{n,k}$ has an invariant S'_{n-1} within which it determines an isomorphic $g'_{n,k}$ of linear transformations whose coefficients are rational numbers. The importance of the finite types in these series of groups would seem to indicate that interesting applications of the infinite types may be expected.

A definite application of the foregoing theory to the determination of the lines on a general cubic surface K^3 will be made.[6] As has been mentioned earlier, cubic curves on P_6^2 map S_2 upon a general cubic surface which appears in Cremona's hexahedral form, C^3. The rational invariants of P_6^2 are rational invariants of the surface after the adjunction of the irrationality which isolates a set of six skew lines of a double six. Such invariants determine, and are determined by, the invariant spreads of $G_{6!}$ in Σ_4. If it be required further that the spreads be invariant under the extended group, $G_{6,2}$ of order 51840 in Σ_4 they determine the invariants of C^3 itself. In this way not only the invariants but also the linear covariants of C^3 are calculated and identified with those of K^3. Now K^3 being given, its invariants and linear covariants are known. The invariants are the known quantities in the form-problem of $G_{6,2}$. The solution of this form-problem furnishes a point P in Σ_4, the map of P_6^2. From P_6^2 as indicated earlier the lines of C^3 as well as the linear covariants of C^3 are derived. The linear transformation obtained from the identification of the linear covariants of C^3 and K^3 transforms the known lines of C^3 into the required lines of K^3. All of the processes outlined here are rational except the solution of the form-problem of $G_{6,2}$. This may be dismissed as a problem in the theory of functions or it may by means of an accessory irrationality be reduced to the form-problem which occurs in the trisection of the periods of the hyperelliptic functions of genus two. The ultimate importance of this latter problem has been pointed out by Klein.[7] It would be more desirable however to utilize directly the theta-modular functions in the solution of the problem of $G_{6,2}$ as has been done in the case of the sextic.

The developments sketched above suggest a number of inquiries which promise results of interest in various directions. Also some of the problems considered such as that of the invariants of P_n^k are worthy of closer study than has been given them in the account here reported.

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[1] This investigation has been pursued under the auspices of the Carnegie Institution of Washington, D. C.

[2] For references see Study, *Math. Ann.*, *Leipzig*, **60**, 348 (footnote).

[3] E. H. Moore, *Amer. J. Math.*, **22**, 279 (1900).

[4] *Trans. Amer. Math. Soc.*, **9**, 396 (1908), and **12**, 311 (1911). The relation of these papers to the earlier work of Klein and others is there set forth at length.

[5] S. Kantor has used this device for an S_2 in his crowned memoir: *Premiers Fondements pour une Théorie des Transformations Périodiques Univoques*, Naples (De Rubertis), 1891.

[6] This will appear as a joint paper by Mr. C. P. Sousley and A. B. Coble.

[7] Letter to Hermite, *J. Math.*, *Paris*, Ser. 4, **4**, 169. See also *Math. Ann.*, *Leipzig*, Witting, **29**, 167; Maschke, **33**, 317, **36**, 190; and Burkhardt, **38**, 161.

THE STRAIGHT LINES ON MODULAR CUBIC SURFACES

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1. In ordinary space a cubic surface without singular points contains exactly 27 straight lines, of which 27, 15, 7, or 3 are real; there are 45 sets of three coplanar lines, the three of no set being concurrent. In modular space, in which the coördinates of points and the coefficients of the equations of lines or surfaces are integers or Galois imaginaries taken modulo 2, it is interesting to notice that *three coplanar lines on a cubic surface may be concurrent* (§2). A point with integral coördinates is called real. A line or surface is called real if the coefficients of its equations are integers. In space with modulus 2, the number of real straight lines on a cubic surface without singular points is 15, 9, 5, 3, 2, 1, or 0.

We shall give here an elementary, self-contained, investigation of some of the most interesting cubic surfaces modulo 2. A complete classification of all such surfaces under real linear transformation will appear in the *Annals of Mathematics*, but without the present investigation of the configuration of their lines.

2. Every real point of space modulo 2 is on the surface[1]

$$xy(x+y) = zw(z+w). \tag{1}$$

since each member is an even integer when x, y, z, w are integers.